Forced Response of an Arbitrary Number of Electromagnetic Coupled Resonators

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***Abstract*—Coupled resonators appear as the building blocks of many systems and devices. Recently, Energy Coupled Mode Theory (ECMT), a general coupled mode formalism in an eigenvalue problem form, was introduced to estimate the frequencies and fields of an arbitrary number of coupled resonators. In the current article, impressed sources are introduced to generalize ECMT to cover cases where arbitrary input excitations are applied. The system is represented by a matrix transfer function in the complex domain. To demonstrate how the transfer function can be applied, the efficiency of a classical system of two inductively coupled resonators is calculated under different load values and input frequencies.**

I. INTRODUCTION

Coupled resonators (CRs) arise as the building blocks of different devices and systems. For example they are the unit cells in resonant type metamaterials, magneto-inductive waves [1] and wireless power transfer systems [2].

Usually, CRs are analyzed using either circuit models or coupled mode theory [2], [3]. In both approaches the coupling coefficient κ and other on diagonal terms are known a priori, where they are calculated using energy considerations. Energy Coupled Mode Theory (ECMT), an eigenvalue problem, was introduced to estimate the coupled frequencies and fields. It can be considered the EM analog of Molecular Orbital Theory [4]. ECMT was used to determine the coupled modes in Electron Spin Resonance Probes [5] and the behaviour of resonators in the presence of conducting planes [6]. Moreover, it was used to derive general expressions of κ [7].

In the current article, a set of second order coupled differential equations in the modes amplitudes is derived to determine the response of an arbitrary number of resonators to arbitrary inputs. The set of equations reduces to the ECMT eigenvalue problem form in the absence of input excitations. To illustrate how the derived equations can be used, the transfer efficiency of a classical two inductively coupled resonant coils is calculated for different load values and input frequencies.

II. ANALYSIS

The $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of CRs are expanded in terms of the uncoupled ones ($\boldsymbol{E}_k$ and $\boldsymbol{H}_k$). Here, the expansion coefficients $a_k(t)$ and $b_k(t)$ are time dependent to take into account the effect of the impressed source ($\boldsymbol{J}_{\text{imp}}$) which excites the system. Accordingly,

$$\boldsymbol{E} = \sum_{k=1}^{N} a_k(t)\,\boldsymbol{E}_k \text{ and } \boldsymbol{H} = \sum_{k=1}^{N} b_k(t)\,\boldsymbol{H}_k, \tag{1}$$

where N is the number of interacting modes. As was carried out in [4], the coupled mode equations can be determined by expanding $\nabla\cdot(\boldsymbol{E}_k^*\times\boldsymbol{H})$ and $\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}_k^*)$, using Maxwell's equations and (1). The time dependency of $a_k(t)$ and $b_k(t)$, and the presence of $\boldsymbol{J}_{\text{imp}}$ generalize the coupled equations to

$$\mathcal{A}\dot{\boldsymbol{a}} + (\mathcal{M}+\mathcal{F}-i\Omega\mathcal{B})\,\boldsymbol{b} = \mathcal{J} \tag{2}$$

and

$$\mathcal{G}\dot{\boldsymbol{b}} + \left(\mathcal{M}^\dagger - i\Omega\mathcal{D}\right)\boldsymbol{a} = 0, \tag{3}$$

where $\mathcal{A}$, $\mathcal{M}$, $\mathcal{F}$, Ω, $\mathcal{B}$, $\mathcal{G}$ and D are the overlap integrals and are defined in [4]. $\mathcal{J}$ is an $N\times 1$ column vector which includes the forcing terms due to the interaction of the modes with $\boldsymbol{J}_{\text{imp}}$. Its k^{th} row is given by

$$\mathcal{J}_k = -\int_V \boldsymbol{J}_{\text{imp}}\cdot\boldsymbol{E}_k^*\;dv. \tag{4}$$

Eqs. (2) and (3) are coupled differential equations in the fields amplitudes. They are the generalized coupled mode equations which in the absence of $\boldsymbol{J}_{\text{imp}}$ reduces to the ones obtained in [4]. A magnetic source can be easily included, where its effect appears as a forcing term in the R.H.S of (3). Taking the time derivative of (2) and substituting for $\dot{\boldsymbol{b}}$ from (3), one finds

$$\ddot{\boldsymbol{a}} - \mathcal{A}^{-1}(\mathcal{M}+\mathcal{F}-i\Omega\mathcal{B})\,\mathcal{G}^{-1}\left(\mathcal{M}^\dagger - i\Omega\mathcal{D}\right)\boldsymbol{a} = \mathcal{A}^{-1}\dot{\mathcal{J}}. \tag{5}$$

For small radiation losses and thin conductors, (5) reduces to

$$\ddot{\boldsymbol{a}} + \mathcal{K}\boldsymbol{a} = \mathcal{A}^{-1}\dot{\mathcal{J}}, \tag{6}$$

where $\mathcal{K} = \mathcal{A}^{-1}\mathcal{D}^\dagger\Omega\mathcal{G}^{-1}\Omega\mathcal{D}$ [4]. Losses are taken into account by replacing ω_k in Ω by $\omega_k + i|\sigma_k|$, where $-|\sigma_k|$ is the decay rate of the k^{th} mode.

It is worth noting that unlike most of coupled mode formalisms, κ and on-diagonal terms are determined directly from Eq. (6) [2], [4]. Moreover, the effect of nearby conducting planes (PEC or PMC) can be taken into account by considering the coupling with the induced image resonators [6].

Taking the Laplace transform of (6), the response can be expressed in the $s-$domain as

$$\mathcal{L}\{\boldsymbol{a}\} = \boldsymbol{\Phi}(s)\left[s\dot{\boldsymbol{a}}(0) + \boldsymbol{a}(0)\right] + s\boldsymbol{\Phi}(s)\,\mathcal{L}\{\mathcal{J}\}, \tag{7}$$

where $\boldsymbol{\Phi}(s) \equiv \left(s^2\boldsymbol{I}+\mathcal{K}\right)^{-1}$is the $N\times N$ matrix transfer function. When the system is excited by sinusoidal inputs of frequency ω, the steady state response is determined by the last term in the R.H.S of (7) as

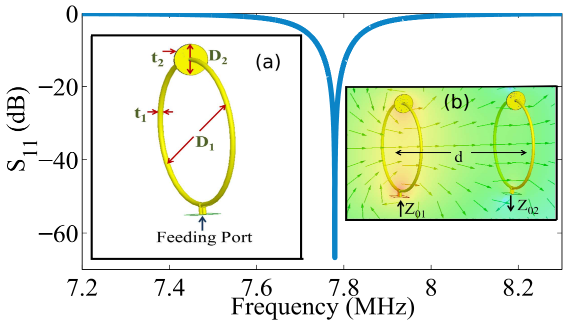

Fig. 1. The Reflection Coefficient of a capacitively loaded coil having a resonance frequency $f_0 \approx 7.79$ MHz. Inset (a) A single coil. Inset (b): Magnetic field of the CRs at resonance. Dimensions: D_1= 62 cm, t_1= 2 cm, D_2= 13.26 cm, dielectric disc thickness, t_2= 2 mm, d= 80 cm and dielectric constant= 10. $\tan\delta$ of dielectric was set to 1/8000 (scenario 1) and 1/80 (scenario 2).

$$\tilde{\boldsymbol{a}}\left(\omega\right)=i\omega\boldsymbol{\Phi}\left(\omega\right)\mathcal{L}\{\mathcal{J}\}|_{s=i\omega}, \tag{8}$$

where s was replaced by $i\omega$. For many practical cases, the system performance can be evaluated without the need of the exact $\mathcal{J}$ terms values. The following example illustrates how (8) can be used to evaluate the transfer efficiency of two inductively CRs.

A. Resonant Inductively Coupled Coils

To demonstrate how (8) can be applied, a system of two capacitively loaded coils which are inductively coupled is analyzed. Finite Element simulation (HFSS, Ansys Corporation, Pittsburgh, PA, USA) is used to verify the results. The coils have the same dimensions of those in [2]. For completeness, the dimensions and relevant parameters are presented in the caption of Fig. 1. The Fig. also shows the simulated reflection coefficient S_{11} which indicates that the resonant frequency $f_0 \approx 7.8$ MHz. Based on the dielectric disks loss tangent, two scenarios are considered. The dielectric loss tangent is set to 1/8000 (scenario 1) and 1/80 (scenario 2), respectively. The two coils are positioned 80 cm apart. A sinusoidal input is applied to the excitation port of the source coil (hereafter denoted by the subscript 1). The transfer efficiency η can be calculated after $|\tilde{a}_2/\tilde{a}_1|$ is determined. This ratio depends on the 2×2 transfer matrix $\boldsymbol{\Phi}\left(\omega\right)$. The coupling is primarily magnetic, thus the off-diagonal terms of $\mathcal{D}$ and $\mathcal{A}$ vanish. Moreover, $\boldsymbol{J}_{\text{imp}}$ is only coupled to the first resonator. The coupling coefficient κ is the main factor and it is equal to $\mathcal{G}_{12}/\mathcal{G}_{11}$ [4]. Therefore, the quantity $|\tilde{a}_2/\tilde{a}_1|$ is obtained from (8) as

$$\left|\frac{\tilde{a}_2}{\tilde{a}_1}\right|=\left|\frac{\kappa\left(\omega_0+i\sigma_0\right)\left(\omega_0+i\sigma_0+i\sigma_w\right)}{\left(\omega_0-\omega+i\sigma_0+i\sigma_w\right)\left(\omega_0+\omega+i\sigma_0+i\sigma_w\right)}\right|, \tag{9}$$

where σ_0 and σ_w are the intrinsic decay rate of the coils and load, respectively. Once the ratio of the two amplitudes is known, η can be found [2]

$$\eta=100\frac{\sigma_w|\tilde{a}_2/\tilde{a}_1|^2}{\left(\sigma_w+\sigma_0\right)|\tilde{a}_2/\tilde{a}_1|^2+\sigma_0}. \tag{10}$$

On the other hand η, as a function of σ_w, can be determined from the S parameters as

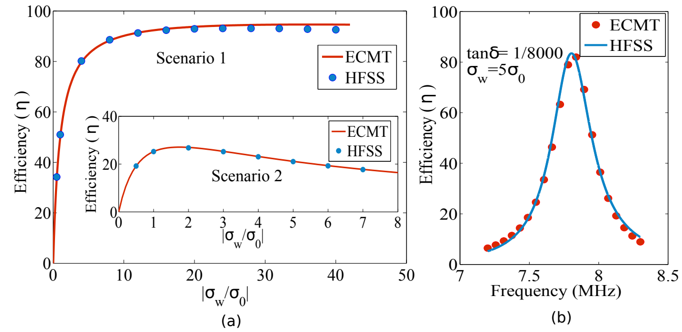

Fig. 2. The efficiency of the coupled system calculated using the Forced ECMT equation of motion and Finite Element simulation when (a) $\tan\delta = 1/8000$ and (inset) $\tan\delta = 1/80$ and $f = f_0$. (b) $\tan\delta = 1/8000$, $f = 7.2 - 8.3$ MHz and $\sigma_w/\sigma_0 = 5$.

$$\eta\left(\sigma_w\right)=100\frac{|S_{21}|^2}{1-|S_{11}|^2}. \tag{11}$$

The characteristic impedance (Z_{01}) of the feeding line was chosen to be much smaller than the coil intrinsic resistance R_0 ($Z_{01} = R_0/100$) and thus does not load the structure. σ_w was controlled by changing the characteristic impedance of the output port (Z_{02}) such that the $\sigma_w = \sigma_0 Z_{02}/R_0$. Fig. 2(a) shows η as a function of σ_w, where the excitation frequency is fixed at f_0. ECMT gives accurate results even when the resonant coils are lossy ($Q_0 \approx 80$). On the other, Fig. 2(b) presents the results when σ_w was held at σ_0 and the input excitation frequency f was swept over a 20% window. Again, ECMT agrees with the finite element simulations as long as $|f-f_0|$ is not very large to the extent that higher order modes, not taken into account, are excited. In this article, we derived a matrix transfer function in the complex plane to describe the performance of an arbitrary number of CRs. We demonstrated how it can be applied to calculate the efficiency of a classical system of two inductively CRs. Nevertheless, the matrix transfer function can be used to calculate performance metrics of an arbitrary number of CRs in a complex environment defined by the presence of conducting planes.

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